MATH 161 FINAL EXAM SOLUTIONS

MARCH 16, 2011

1. Let A be any set. Prove that there is no surjective map from A to $\mathcal{P}(A)$.

Solution: Suppose $F : A \to \mathcal{P}(A)$. Let $S = \{x \in A : x \notin F(x)\}$. Then for every $x \in A$, $x \in S$ if and only if $x \notin F(x)$. Thus $S \neq F(x)$, since x is an element of one but not the other. Thus S is not in the range of F, so F is not surjective. □

2. State the Cantor-Bernstein Theorem in terms of functions (i.e, without using the word "cardinality", the symbol $|\cdot|$, etc.).

Solution: If there is an injective (or one-to-one) map $F : A \to B$ and an injective map $G : B \to A$, then there is a bijection from A to B.

3. Let A and B be sets with $A \neq \emptyset$. Prove that there is an injective mapping $F : A \to B$ if and only if there is a surjective mapping $G : B \to A$.

Solution: Suppose there is an injective map $F : A \to B$. Let a be an element in A and let C be the range of A. Note that $F^{-1} : C \to A$ is a surjective function. Thus

$$G(x) = \begin{cases} F^{-1}(x) & \text{if } x \in C \\ a & \text{if } x \in B \setminus C \end{cases}$$

defines a surjective function from B to A.

[Alternate way of saying the same thing: let $G = F^{-1} \cup ((B \setminus C) \times \{a\})$.]

Conversely, suppose $G : B \to A$ is surjective.

By the axiom of choice, there is a function $\phi : \mathcal{P}(B) \to B$ such that $\phi(S) \in S$ for every nonempty $S \in \mathcal{P}(B)$.

Now define $F : A \to B$ by

$$F(x) = \phi(\{y \in B : G(y) = x\}).$$

If $x \in A$, then $\{y \in B : G(y) = x\}$ is nonempty since G is surjective. Thus $F(x) \in \{y \in B : G(y) = x\}$, so $G(F(x)) = x$. We have shown: $x \in A$ implies $G(F(x)) = x$. Now suppose x and x' are elements of A with $F(x) = F(x')$. Then $G(F(x)) = G(F(x'))$, so $x = x'$. Thus F is injective. □

4. Recall that addition on $\mathbf{N}$ was defined recursively by

$$a + 0 = a,$$

$$a + S(b) = S(a + b).$$

Prove that $(a + b) + c = a + (b + c)$ for all natural numbers a, b, and c. [You should not assume any other facts, such as commutativity, about addition.]

Solution: We prove it by induction on c. First, it is true for $c = 0$ since

$$(a + b) + 0 = a + b$$

and

$$a + (b + 0) = a + b.$$

Now suppose it is true for $c = k$

$$(a + b) + k = a + (b + k).$$

Taking the successor of both sides gives

$$S((a + b) + k) = S(a + (b + k)).$$

The left side is $(a+b)+S(k)$. The right side is $a+S(b+k)$, which is equal to $a+(b+S(k))$. Thus

$$(a + b) + S(k) = a + (b + S(k)).$$

We have shown that if the assertion holds for $c = k$, then it also holds for $c = S(k)$. Hence by induction, it holds for all c. □

5(a). Let $\mathbf{Q}^+$ be the set of positive rational numbers. Prove that $\mathbf{Q}^+$ is countable by giving a one-to-one map from $\mathbf{Q}^+$ to $\mathbf{N}$.

Solution: Note that for each $r \in \mathbf{Q}^+$, there is a unique pair $m = m(r)$ and $n = n(r)$ of nonzero natural numbers such that (i) $r = m/n$ and (ii) m and n have no common factor (other than 1.)

Then the map

$$F : \mathbf{Q}^+ \to \mathbf{N}$$
$$F(r) = 2^{m(r)} 3^{n(r)}$$

is a one-to-one map (since each nonzero natural number has a unique prime factorization.)

5(b). Let $\mathcal{F}$ be the set of functions $f : \mathbf{N} \to \mathbf{N}$ such that f is eventually 0, i.e., such that $\{n \in \mathbf{N} : f(n) > 0\}$ is finite. Prove that $\mathcal{F}$ is countable by giving a one-to-one map from $\mathcal{F}$ to $\mathbf{N}$.

Solution: For $f \in \mathcal{F}$, let $S(f) = \{n \in \mathbf{N} : f(n) \neq 0\}$. Let $p : \mathbf{N} \to \{\text{prime numbers}\}$ be a bijection. (We might as well choose the bijection such that $p_0 < p_2 < p_3 < \dots$, so that $p_0 = 2$, $p_1 = 3$, etc.) Define $\phi : \mathcal{F} \to \mathbf{N}$ by

$$\phi(f) = \Pi_{n \in S(f)} (p_n)^{f(n)}.$$

Then ϕ is a one-to-one map (since each nonzero natural number has a unique prime factorization.)

6. (Cardinal arithmetic.) In this problem, you may use the fact that $\kappa \cdot \kappa = \kappa$ for every infinite cardinal κ.

(a). Prove that if κ and λ are cardinals with κ infinite and with $1 \leq \lambda \leq \kappa$, then $\kappa \cdot \lambda = \kappa$.

Proof. Multiply $1 \leq \lambda \leq \kappa$ by κ to get

$$\kappa \leq \kappa \cdot \lambda \leq \kappa \cdot \kappa = \kappa.$$

Thus $\lambda \cdot \kappa = \kappa$. [Remark: we are using the Cantor-Bernstein Theorem here. Since $\kappa \leq \lambda \cdot \kappa$ and since $\lambda \cdot \kappa \leq \kappa$, in fact they must be equal.] □

(b). Prove that if κ and λ are cardinals with $0 \leq \lambda \leq \kappa$ and with κ infinite, then $\kappa + \lambda = \kappa$.

Proof. Adding κ to $0 \leq \lambda \leq \kappa$ gives

(*) $$\kappa \leq \kappa + \lambda \leq \kappa + \kappa = \kappa \cdot 2$$

The right hand side is equal to κ by part (a). Thus we must have equality in (*). □

(c). Prove that if κ is an infinite cardinal and if λ is a cardinal such that $2 \le \lambda \le 2^\kappa$, then $\lambda^\kappa = 2^\kappa$.

Proof. Since $2 \le \lambda \le 2^\kappa$,

$$2^\kappa \le \lambda^\kappa \le (2^\kappa)^\kappa = 2^{\kappa\cdot\kappa} = 2^\kappa.$$

By Cantor-Bernstein, we must have equality. $\square$

7(a). Let $\alpha > 0$ be an ordinal and let a and b be nonzero natural numbers. Find the Cantor Normal Form for the ordinal $\gamma = (\omega^\alpha a + b)^2$.

Solution (using results from hw problems):

$$\begin{aligned}(\omega^\alpha a + b)^2 &= (\omega^\alpha a + b)(\omega^\alpha a + b)\\ &= (\omega^\alpha a + b)\omega^\alpha a + (\omega^\alpha a + b)b\\ &= \omega^{\alpha+\alpha} a + (\omega^\alpha ab + b)\\ &= \boxed{\omega^{\alpha\cdot 2} a + \omega^\alpha ab + b}.\end{aligned}$$

7(b). Let $\alpha > \beta > 0$ be ordinals and let a and b be nonzero natural numbers. Find the Cantor Normal Form for the ordinal $\eta = (\omega^\alpha a + \omega^\beta b)^2$.

Solution:

$$\begin{aligned}(\omega^\alpha a + \omega^\beta b)^2 &= (\omega^\alpha a + \omega^\beta b)(\omega^\alpha a + \omega^\beta b)\\ &= (\omega^\alpha a + \omega^\beta b)\omega^\alpha a + (\omega^\alpha a + \omega^\beta b)\omega^\beta b\\ &= \omega^{\alpha+\alpha} a + \omega^{\alpha+\beta} b\\ &= \boxed{\omega^{\alpha\cdot 2} a + \omega^{\alpha+\beta} b}.\end{aligned}$$

7(c). Find the Cantor Normal Form for the ordinal $\delta = (\omega + 3)^{(\omega+7)}$.

Solution: I claim that

(*) $$(\omega+3)^n = \omega^n + \omega^{n-1}\cdot 3 + \cdots + \omega\cdot 3 + 3 \text{ for } n \in \mathbf{N}^+.$$

The proof is by induction on n. It is trivially true for $n = 1$. Assuming it is true for n, multiply both sides of (*) on the right to get

$$\begin{aligned}(\omega+3)^{n+1} &= (\omega^n + \omega^{n-1}\cdot 3 + \cdots + \omega\cdot 3 + 3)\cdot(\omega+3)\\ &= (\omega^n + \omega^{n-1}\cdot 3 + \cdots + \omega\cdot 3 + 3)\cdot\omega + (\omega^n + \omega^{n-1}\cdot 3 + \cdots + \omega\cdot 3 + 3)\cdot 3\\ &= \omega^{n+1} + (\omega^n\cdot 3 + \omega^{n-1}\cdot 3 + \cdots + \omega\cdot 3 + 3),\end{aligned}$$

which is (*) with n replaced by $(n+1)$. This completes the proof by induction of (*).

Taking the supremum of both sides of (*) over $n \in \mathbf{N}$ gives

(†) $$(\omega+3)^\omega = \omega^\omega.$$

Hence by (*) and (†),

$$\begin{aligned}(\omega+3)^{\omega+7} &= (\omega+3)^\omega\cdot(\omega+3)^7\\ &= \omega^\omega\cdot(\omega^7 + \omega^6\cdot 3 + \omega^5\cdot 3 + \omega^4\cdot 3 + \omega^3\cdot 3 + \omega^2\cdot 3 + \omega\cdot 3 + 3)\\ &= \boxed{\omega^{\omega+7} + \omega^{\omega+6}\cdot 3 + \omega^{\omega+5}\cdot 3 + \omega^{\omega+4}\cdot 3 + \omega^{\omega+2}\cdot 3 + \omega^{\omega+1}\cdot 3 + \omega^\omega\cdot 3}.\end{aligned}$$

8. Let A be a set and let R be a symmetric relation on A. Prove that there is a set $S \subset A$ such that

(1) If $x \in S$ and $y \in S$ and $x \neq y$, then $(x, y) \in R$.
(2) If $S \subsetneq T \subset A$, then there are elements $x \in T$ and $y \in T$ such that $x \neq y$ and $(x, y) \notin R$.

Solution1 (transfinite recursion): Let $<$ be a well-ordering of A. Define a function $F : A \to \{0, 1\}$ by transfinite recursion as follows:

$$F(x) = \begin{cases} 1 & \text{if } xRy \text{ for every } y < x \text{ such that } F(y) = 1, \\ 0 & \text{if not.} \end{cases}$$

Now let $S = \{x \in A : F(x) = 1\}$.

Suppose $x, y \in S$ with $x \neq y$. Then one of x and y, say y, must be smaller than the other. Since $F(y) = 1$, $y < x$, and $F(x) = 1$, it follows that xRy (by definition of x) and therefore that yRx also (by symmetry). This proves (1).

Now suppose $S \subsetneq T \subset A$. Then $T \setminus S$ is nonempty, so it has a least element x. Since $x \notin S$, $F(x) = 0$, which means there is a $y < x$ such that $F(y) = 1$ and such that $(x, y) \notin R$. Since $F(y) = 1$, $y \in S$ and therefore $y \in T$. □

9(a). Complete the definition: a well-ordering of a set A is

a linear ordering of A such that every nonempty subset of A has a least element.

Note: "a partial ordering of A such that every nonempty subset of A has a least element" is also correct. [It's equivalent to the usual definition.]

9(b). Suppose that $(W, <)$ is a well-ordered set and that $F : W \to W$ is an order-preserving mapping, i.e., a mapping such that $x < y$ implies that $F(x) < F(y)$. Prove that $x \leq F(x)$ for every $x \in W$.

Proof. Suppose it's false. Then there is a smallest element a of W for which a is not $\leq F(a)$, i.e., for which $F(a) < a$. Let $b = F(a)$. Then $b < a$, and from $F(a) < a$, it follows that $F(F(a)) < F(a)$, i.e, that $F(b) < b$. But that contradicts the choice of a. □

9(c). For $x \in W$, let $W[x] = \{y \in W : y < x\}$. Let a and b be elements of W. Prove that if $(W[a], <)$ and $(W[b], <)$ are isomorphic, then $a = b$.

Proof. Suppose $b < a$. Let $F : W[a] \to W[b]$ be an isomorphism. Then F is an order-preserving map from $W[a]$ into $W[a]$, so by part (a) (applied with $W[a]$ in place of W), $x \in F(x)$ for every $x \in W[a]$. In particular, $b \leq F(b)$. But that proves that $F(b) \notin W[b]$, a contradiction (since $F : W[a] \to W[b]$.)

In exactly the same way, we get a contradiction if $a < b$.

Since a is not less than b and b is not less than a, in fact $a = b$. □

10. Let $A_i (i \in I)$ and $B_i (i \in I)$ be families of sets indexed by the same set I. Suppose that $|A_i| < |B_i|$ for every $i \in I$. **König's Theorem** states that

$$|\cup_{i \in I} A_i| < |\Pi_{i \in I} B_i|.$$

In this problem you will (if all goes well) prove König's theorem. Let $U = \cup_i A_i$ and $P = \Pi_{i \in I} B_i$.

10(a). Consider any map $F : U \to P$. Let

$$C_i = \{h(i) : h = F(a) \text{ for some } a \in A_i\}.$$

Let $D_i = B_i \setminus C_i$. Explain why D_i must be nonempty.

Proof. For each i, consider the map

$$\begin{aligned} &\phi_i : A_i \to C_i \\ &\phi_i(a) = F(a)(i) \quad \text{or } F(a)_i, \text{ if you prefer)} \end{aligned}$$

Since $|A_i| < |B_i|$, there is no surjective map from A_i to B_i. Thus C_i (which is the image of A_i under ϕ_i) is not equal to B_i, or, equivalently, $B_i \setminus C_i$ is nonempty. □

10(b). Explain why $\Pi_{i \in I} D_i$ must be nonempty.

Solution: This is a version of the axiom of choice: if $D_i (i \in I)$ is an index family of nonempty sets, then $\Pi_{i \in I} D_i$ is nonempty.

10(c). Explain how König's Theorem follows from parts (a) and (b).

Solution: Suppose $F : \cup_i A_i \to \Pi_i B_i$ as in part (a). We claim that F is **not** surjective. For let $g \in \Pi_i D_i$ as in parts (a) and (b). (Such a g exists by part (b).) I claim that g is not in the range of F. To see this, suppose $a \in \cup_i A_i$. Then $a \in A_i$ for some i. Then $F(a)_i \in C_i$, but $g_i \in D_i = B_i \setminus C_i$, so $F(a)_i \neq g_i$. Therefore $F(a) \neq g$.

We have shown that $g \neq F(a)$ for every $a \in \cup_i A_i$. This proves that g is not in the range of F, so F is not surjective.

Also, $\Pi_{i \in I} B_i$ is nonempty (by the axiom of choice). Thus (by problem 3) there is no injective map from $\Pi_i B_i$ to $\cup_i A_i$. Thus $|\Pi_i B_i|$ is not less than $|\cup_i A_i|$, so $|\cup_i A_i| < |\Pi_i B_i|$.

10(d). Let $A_n \, (n \in \mathbf{N})$ be subsets of $\mathbf{R}$ such that for each n, $|A_n| < \mathbf{R}$. Using König's Theorem, prove that $\mathbf{R} \neq \cup_{n \in \mathbf{N}} A_n$. (Thus $\mathbf{R}$ cannot be the union of countably many subsets each of which has cardinality $< |\mathbf{R}|$. It follows that $|\mathbf{R}|$ cannot be $\aleph_\omega$, since $\aleph_\omega$ is the union of countably many smaller cardinals: $\aleph_\omega = \cup_{n \in \mathbf{N}} \aleph_n$.)

Solution: By König,

$$\begin{aligned} |\cup_{n \in \mathbf{N}} A_n| &< |\Pi_{n \in \mathbf{N}} A_n| \\ &\leq |\Pi_{n \in \mathbf{N}} \mathbf{R}| \qquad (\text{because } \Pi_{n \in \mathbf{N}} A_n \subset \Pi_{n \in \mathbf{N}} \mathbf{R}) \\ &= |\mathbf{R}^{\mathbf{N}}| = |\mathbf{R}|. \end{aligned}$$

Thus $|\cup_{n \in \mathbf{N}} A_n| < |\mathbf{R}|$, so $\cup_n A_n \neq \mathbf{R}$. □

Bonus Problem (optional). Let $F : \omega_1 \to \omega_1$ be a map such that $F(x) \leq x$ for every $x \in \omega_1$ and such that $\{y : F(y) = x\}$ is at most countable for every $x \in \omega_1$.

Let C be an unbounded subset of ω_1 with the following properties:

(1) C is unbounded (or, equivalently, uncountable), and
(2) C is closed, meaning that if S is an nonempty, bounded subset of C, then $\sup S$ is also in C.

Prove that $\{x \in C : F(x) = x\}$ is nonempty.

[**Remark**: it follows that $\{x \in C : F(x) = x\}$ is unbounded. For if $\alpha \in \omega_1$, then the set $C' = \{x \in C : x \geq \alpha\}$ is also closed and unbounded and therefore (if the problem is correct) $\{x \in C' : F(x) = x\}$ is nonempty. In other words, for every $\alpha \in \omega_1$, the set $\{x \in C : F(x) = x\}$ has elements $\geq \alpha$. Thus $\{x \in C : F(x) = x\}$ is unbounded.]

[Hint: Prove it by contradiction. Suppose there is such a function F. Use that function to define a new function that contradicts a theorem we proved in the lecture about the apple game. Or, show how you could use F to devise a strategy for the apple game in which you never get fined.]

Solution: Since $C \subset \omega_1$ is closed, it has the following property:

(*) If C contains any element $\leq \alpha$, it contains a greatest such element.

Suppose there exist an $F : \omega_1 \to \omega_1$ such that $F(x) \leq x$ for all x, such that $\{y : F(y) = x\}$ is at most countable for every $x \in \omega_1$, and such that $F(x) < x$ for all $x \in C$.

Define a new function G as follows:

(1) If $x \in C$, we let $G(x) = F(x)$.
(2) If $x \notin C$ and if there is an element of C less than x, we let $G(x)$ be the largest element of C that is $< x$.
(3) If $x \notin C$ and if there is no element of C less than x, then we let $G(x)$ be 0.

From (2) and (3), we deduce

(†) If $\alpha \in C$, if $x \notin C$, and if $x > \alpha$, then $G(x) \geq \alpha$.

I claim that if $x > 0$, then $G(x) < x$. If $x \in C$, this is because $G(x) = F(x) < x$. If $x \notin C$ and $x > 0$, then $F(x) < x$ by (2) or (3) above.

I also claim that for each x, $\{y : G(y) = x\}$ is countable. Too see this, let α be an element of C that is greater than x. (Such an α exists since C is unbounded.) Then

$$\begin{aligned} \{y : G(y) = x\} &= \{y \in C : G(y) = x\} \cup \{y \notin C : G(y) = x\} \\ &= \{y \in C : F(y) = x\} \cup \{y \notin C : G(y) = x\} \\ &\subset \{y : F(y) = x\} \cup \{y : y < \alpha\} \\ &= \{y : F(y) = x\} \cup \alpha. \end{aligned}$$

Now $\{y : F(y) = x\}$ is at most countable by hypothesis, and α is at most countable since $\alpha < \omega_1$.

However, we proved in class that no such G exists.

[If you don't remember that theorem, note that such a G would let us win the apple game. At each time $\alpha > 1$, you discard one of the apples that you received at time $G(\alpha)$. Note that in this way you never run out of apples. In fact, if you number the apples you receive at time 0, you can easily arrange to discard only the even numbered ones of the apples, so that you will have infinitely many left over at the end of the game.]