MATH 161 HOMEWORK 2

Note: As mentioned in class, I use "$A \subset B$" to mean "A is a subset of B" (and "$A \subsetneq B$" to mean "$A \subset B$ and $A \neq B$").

1. Let R be a relation. Let $S = R \cup R^{-1}$. Prove carefully (i.e., justifying each step) that (i) S is a symmetric relation, and (ii) if R is a subset of a symmetric relation T, then S is a subset of T.

Remark: Of course R is a subset of S, so (i) and (ii) mean that (in a sense) S is the smallest symmetric relation that contains R as a subset.

2. Suppose F is a nonempty set of transitive relations. (That is, suppose that F is nonempty and that each element of F is a transitive relation.) Prove that

$$\cap F$$

(i.e., $\cap_{R \in F} R$) is a transitive relation. (Remark: this is also true if the word "transitive" is replaced by "symmetric" or by "reflexive" or by "equivalence".)

3. Suppose that R is a relation. Prove there is a transitive relation T such that $R \subset T$. (There are actually many such T's.)

4. Suppose R and E are relations. We say that E **extends** R provided the following is true: xRy implies xEy for all x and y. [In other words, "E extends R" is just another way of saying "$R \subset E$".] Suppose R is a relation. Prove that there is a transitive relation T with the following properties:

(1) T extends R.
(2) If $\tilde{T}$ is a transitive relation that extends R, then $\tilde{T}$ extends T.

[Note that T is, in a sense, the smallest transitive relation that extends R.]

5. A set A is called a **transitive** set provided each element of A is also a subset of A. The successor $S(A)$ of a set A is the union of A and $\{A\}$:

$$S(A) = A \cup \{A\}.$$

Suppose A is a transitive set. Prove that $S(A)$ is also a transitive set.

Warning: don't confuse "transitive set" with "transitive relation".

6. The axiom of foundation (which we will introduce later) says that if x is any non-empty set, then there is an element y of x that is disjoint from x. (That is, there is an element y of x such that x and y have no elements in common.) Using the axiom of foundation, prove that it is impossible to have sets a and b that are elements of each other. (In other words, if a is an element of b, then b cannot be an element of a.)

7. Let x and y be any sets. Suppose $S(x) = S(y)$. Using the axiom of foundation, prove that $x = y$. (Here $S(x)$ is the successor of x, as in problem 5.)

8. Mr. Zanzibar Buck-Buck McFate knows about the positive integers $1, 2, 3, \dots$ but knows nothing about set theory. One day he sees a list of some of the axioms of set theory (see the list below). However, he thinks that "x is a set" means "x is a positive integer" and that "x is an element of y" means "x is a factor of y" (i.e., y is divisible by x). According to his interpretation of "set" and "element", which axioms turn out to be true and which turn out to be false?

Note: Zanzibar's list contains the weak forms of the axioms, namely:

Existence: there exists a set.
Extension: as in our text.
Selection (i.e., comprehension): as in our text.
Pair: for all sets x and y, there exists a set z such that $x \in z$ and $y \in z$.
Union: For any set z, there is a set u such that $x \in y$ and $y \in z$ implies $x \in u$.
Power set: For every set A, there is a set P such that every subset of A is an element of P.
Foundation: Every nonempty set x has an element y disjoint from x.
Infinity: there is a set I with the following two properties: (i) the emptyset is an element of I, and (ii) for every element x of I, the successor $S(x)$ of x is also an element of I.

Example: The axiom of extension says (to Zanzibar) that two positive integers with the same factors are equal. That is true. (Proof: the largest factor of x is x itself. If x and y have the same factors, they have the same largest factor, and are therefore equal.)

Note: Zanzibar looks up the definitions of "subset", "nonempty", disjoint", "successor", and "emptset", and of course interprets them according to his understanding of "set" and "element". Thus for example "A is a subset of B" means to Zanzibar that every factor of A is also a factor of B.