**MATH 161 HOMEWORK 3 (DUE THUR, JAN 27 IN CLASS)**

If $A$ and $B$ are sets, then $B^A$ denotes the set of all functions $f : A \to B$:

$$\begin{aligned} B^A &= \{f \subset A \times B : f \text{ is a function}\} \\ &= \{f \in \mathcal{P}(A \times B) : f \text{ is a function}\}. \end{aligned}$$

(This notation is introduced in the text, but perhaps was not mentioned in the lectures.)

0. (To do, but not turn in.) In class, we proved that $\mathbf{N}$ is a transitive set and that $\in$ defines a strict linear ordering on $\mathbf{N}$. We also prove that if $n \in \mathbf{N}$, then $n$ is a transitive set and that every nonempty subset of $n$ has a least element. Use these facts to prove that every nonempty subset of $\mathbf{N}$ has a least element. [Remark: this is an extremely important and useful fact.]

1. Prove that $a + (b + c) = (a + b) + c$ for all natural numbers $a$, $b$, and $c$.

2. Suppose that $a$ is a natural number and that $a \neq 0$. Prove that $a = S(b)$ for some natural number $b$.

3. Prove (without using the axiom of foundation) that if $x$ and $y$ are natural numbers and if $S(x) = S(y)$, then $x = y$.

4. Suppose that $A$ is a transitive subset of $\mathbf{N}$ (the set of natural numbers). Prove that either $A = \mathbf{N}$ or $A$ is an element of $\mathbf{N}$.

5. Suppose that $C$ is a subset of $\mathbf{N}$ with the following property: for every natural number $x$, if $S(x)$ is an element of $C$, then $x$ is an element of $C$. Prove that either $C = \mathbf{N}$ or $C$ is an element of $\mathbf{N}$.

6. Prove (without using the axiom of foundation) that there is no function $f : \mathbf{N} \to \mathbf{N}$ such that $f(k + 1) < f(k)$ for all $k \in \mathbf{N}$.

7. Using the axiom of foundation, prove that there is no function $f$ with $\mathrm{dom}(f) = \mathbf{N}$ such that $f(k + 1) \in f(k)$ for all $k \in \mathbf{N}$.

8. Define a relation $<$ on $\mathbf{N}^{\mathbf{N}}$ as follows: $f < g$ if and only if there is a $k \in \mathbf{N}$ such that $f(i) = g(i)$ for all $i < k$ and such that $f(k) < g(k)$.

Prove that $<$ is a strict linear ordering of $\mathbf{N}$. (In other words, prove that $<$ is transitive on $\mathbf{N}$, and that for any $f$ and $g$ in $\mathbf{N}$ with$f$ not equal to $g$, either $f < g$ or $g < f$ (but not both).

9. Define a relation $<$ on $\mathbf{N} \times \mathbf{N}$ as follows: $(a, b) < (c, d)$ if and only if either

$$a < c$$

or

$$a = c \text{ and } b < d.$$

Let $A$ be a nonempty subset of $\mathbf{N} \times \mathbf{N}$. Prove that $A$ has a least element. [Remark: You may use, without proving it, that $<$ is a strict linear ordering of $\mathbf{N} \times \mathbf{N}$. The proof is just like the proof of 8.]

10. (The easy apple game.) You play a game as follows. The game is played in stages. At stage $n$, you are given 10 apples. You must then choose one of your apples (either one of the ten you were just given or one you were given at an earlier stage) and discard it. The game is over after $n$ has run through all the natural numbers. Your object is to have as many apples as possible at the end.

Show that is possible (depending on which apples you choose to discard) for you to end up with no apples, or with infinitely many apples.

(You might think that such a game would never end. But it will end if you play quickly. For instance, if you make your $n$th move in $2^{-n}$ minutes, then the whole game will take only 2 minutes.)