MATH 161 HOMEWORK 4 (DUE THUR, FEB 3 IN CLASS)

1. Prove that $x \leq x + y$ for all natural numbers x and y.

2. Suppose that n and m are natural numbers with $n \leq m$. Prove that there is $k \in \mathbf{N}$ such that $n + k = m$.

3. Let X be a set and F be a set of binary operations on X, i.e., a set of functions $f : X \times X \to X$. Let A be a subset of X and define sets A_n $(n \in \mathbf{N})$ recursively by:

(1) $A_0 = A$.
(2) $A_{n+1} = A_n \cup (\cup_{f \in F} f[A_n \times A_n])$ for $n \in \mathbf{N}$.

Let $U = \cup_{n \in \mathbf{N}} A_n$. Prove that U is closed under the operations in F, i.e., that if $f \in F$, $x \in U$, and $y \in U$, then $f(x, y) \in U$. [An example: $X = \mathbf{R}$ and $F = \{+, \cdot\}$.]

4. Let A, F, and U be as in problem 3. Suppose that $A \subset V \subset X$ and that V is closed under the operations in F. Prove that $U \subset V$. (Thus U is the smallest subset of X that contains A and that is closed under the operations in F.)

5. Let A, F, and U be as in problem 3. Suppose that A and F are both at most countable. Prove that U is at most countable. [For this problem, you may use the fact (which we can't prove until we have the axiom of choice) that a countable union of countable sets is countable (and similarly for "at most countable" in place of "countable". The problem can done without this, but the proof is a little more involved.]

6. A function f with domain $\mathbf{N}$ is called **eventually periodic** if there are natural numbers n_0 and $k \geq 1$ such that $f(n + k) = f(n)$ for all $n \geq n_0$. Let C be the collection of eventually periodic functions $f : \mathbf{N} \to \mathbf{N}$. Prove that C is countable. [For full credit, you should do this without assuming that countable unions of countable sets are countable.]

7. Let X, Y, and Z be sets. Prove that $(X^Y)^Z \equiv X^{Y \times Z}$ by defining a bijection from $(X^Y)^Z$ to $X^{Y \times Z}$. (You don't need to write out the proof that the map you've defined is a bijection.)

8. Mr. Fonebone thinks that the word "set" means "integer" and that "$a \in b$" means "$a + 1 = b$". (Thus to Mr. Fonebone, 5 is an element of 6, but 5 is not an element of 7, for example.) Which of the axioms of set theory are true in Mr. Fonebone's interpretation? Use the weak forms of the axioms (see hw 2). [Suggestion: to avoid confusion, use the terms "F-element", "F-set", "F-subset", etc to refer to Fonebone's interpretation of "element", "set", "subset", etc. Note: we haven't discussed integers yet, but for this problem, you may use anything you know about the integers.]

9. A **fixed point** of a function f is an $x \in \mathrm{dom}(f)$ such that $f(x) = x$. Let A be a set and let $F : \mathcal{P}(A) \to \mathcal{P}(A)$. Suppose that F is **monotone**, i.e., that $X \subset Y \subset A$ implies $F(X) \subset F(Y)$. Prove that F has a fixed point. [Hint: let $T = \{X \subset A : F(X) \subset X\}$. Show that T is nonempty, let $V = \cap T$, and show that V is a fixed point of F.]

10. (A more advanced apple game.) You play a game as follows. At the nth stage, you are given a countably infinite set of rotten apples. You must then choose one of your apples (either one you were just given or one you were given at an earlier stage) and discard it. The game is over after n has run through all the natural numbers. Your object is to have as few apples as possible at the end.

Show that is possible (depending on which apples you choose to discard) for you to end up with no apples.

(Hint: Think of all the apples as labelled by pairs of natural numbers. Apple (p, k) is the pth apple you received at stage k.)