MATH 161 HOMEWORK 5 (DUE THUR, FEB 10 IN CLASS)

1. Let X be a set, let F be a set of binary operations on X, and let A be a subset of X. Assume that A and F are nonempty and at most countable. Thus there is a surjection $n \mapsto f_n$ from $\mathbf{N}^+$ onto F, and a surjection $n \mapsto a_n$ from $\mathbf{N}^+$ onto A. Define a map $g : \mathbf{N} \to X$ recursively as by:

$$g(n) = \begin{cases} a_k & \text{if } n = 2^k \text{ for some } k > 0, \\ f_k(g(p), g(q)) & \text{if } n = 3^k\, 5^p\, 7^q \text{ for some } k > 0,\ p > 0, \text{ and } q > 0, \\ a_1 & \text{for all other } n \in \mathbf{N}. \end{cases}$$

Let $U = \text{range}(g)$. Note that $A \subset U$.

(1) Prove that U is closed under the operations in F.
(2) Prove that if $A \subset V$ and if V is closed under the operations in F, then $U \subset V$.

[Since the range of a function with countable domain is at most countable, this proves that U is at most countable, without using the axiom of choice. See problems 3 and 4 from hw 4.]

2. Let $\mathcal{C}$ be the set of circles in $\mathbf{R}^2$. Prove that $\mathcal{C} \cong \mathbf{R}$ (i.e., that $\mathcal{C}$ and $\mathbf{R}$ have the same cardinality.)

3. Let $\mathcal{D}$ be the collection of circles C in $\mathbf{R}^2$ such that C contains three points with rational coordinates, i.e., three points in $\mathbf{Q}^2$. Prove that $\mathcal{D}$ is countable.

4. If $x \in \mathbf{R}^n$ and $r > 0$, the open ball with center x and radius r is $\mathbf{B}(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}$. A set $U \subset \mathbf{R}^n$ is called **open** provided it has the following property: for each $p \in U$, there is an open ball $B(p, r)$ centered at p such that $\mathbf{B}(p, r) \subset U$. Prove that if $U \subset \mathbf{R}^n$ is an open set, then

$$U = \cup\{\mathbf{B}(x, r) : x \in \mathbf{Q}^n, r \in \mathbf{Q}^+, \mathbf{B}(x, r) \subset U\}.$$

5. Let G be the collection of all open subsets of $\mathbf{R}^n$. Prove that $G \cong \mathbf{R}$.

6. Let $\mathcal{S}$ be the collection of sets V in $\mathbf{R}^2$ such with the following property: V is a union of closed balls with radii $= 1$ and with centers on the x-axis. Prove that $\mathcal{S} \cong \mathcal{P}(\mathbf{R})$. [Compare problems 5 and 6: it makes a big difference whether the balls are open or closed!] (The closed ball in $\mathbf{R}^n$ of radius r and center p is $\{q \in \mathbf{R}^n : |q - p| \le r\}$.)

7. Let $<$ be a (strict) partial order on a set X. We say that $<$ is "well-founded" if every nonempty subset of X has a minimal element. [Be sure you understand the difference between "minimal element" and "least element".] Suppose that $<$ is a well-founded partial order on X. Suppose $f : X \to X$ is a map with the following property: $x < y$ implies $f(x) < f(y)$. Prove that $f(x) \not< x$ for all $x \in X$. In other words, prove that there is no x such that $f(x) < x$.

[Remember that in a partial order, it is possible to have two elements x and y such that $x \neq y$, $x \not< y$ and $y \not< x$. Here's an example of a well-founded partial order that is not linear. Define an order $<$ on $\mathbf{N} \times \mathbf{N}$ as follows: $(a, b) < (c, d)$ if and only if $a < b$ and $c < d$.]

8. Prove that $\mathbf{R}^{\mathbf{R}} \cong \mathcal{P}(\mathbf{R})$. [Hint: use cardinal arithmetic.]

9. Let $<$ be the linear order on $\mathbf{N}^{\mathbf{N}}$ where $f < g$ if and only if there is an $n \in \mathbf{N}$ such that $f(n) < g(n)$ and $f(i) = g(i)$ for all $i < n$. (See hw 3, problem 8.) Find a nonempty subset of $\mathbf{N}^{\mathbf{N}}$ that has no least element.

10*. Let $\mathcal{D}$ be the set of decreasing functions $f : \mathbf{N} \to \mathbf{N}$. (A function $f : \mathbf{N} \to \mathbf{N}$ is called decreasing if $x < y$ implies $f(x) \ge f(y)$ for each $x, y \in \mathbf{N}$.) Let S be a nonempty subset of $\mathcal{D}$. Show that S has a least element (with respect to the order $<$ in problem 9.)