MATH 161 HOMEWORK 7 (DUE THUR, MARCH 3 IN CLASS)

1. Let $<$ be the reverse lexicographic ordering on $\mathbf{N}^4$. (That is, $a < b$ if and only if there is an i such that $a_i < b_i$ and $a_j = b_j$ for all $j > i$.) Find an isomorphism from $(\mathbf{N}^4, <)$ to an ordinal.

2. Use transfinite recursion to define for all ordinals α and β an ordinal $G(\alpha, \beta)$ whose meaning is (roughly speaking) "a β-high stack of exponentiated α's". Thus for example $G(\alpha, 4)$ should be

$$\alpha^{\alpha^{\alpha^{\alpha}}} = \alpha^{\left(\alpha^{(\alpha^{\alpha})}\right)}.$$

3. (a). Prove that if k is a finite ordinal and if α is an infinite ordinal, then $k + \alpha = \alpha$. (b). Suppose that $\alpha < \beta$ and that n and m are nonzero natural numbers. Express $\omega^\alpha \cdot m + \omega^\beta \cdot n$ in Cantor normal form, and prove it. [Remark: this lets you find the Cantor normal form of the sum of any two ordinals written in Cantor normal form.]

4. Suppose $\beta = \omega^{\alpha_1} \cdot k_1 + \omega^{\alpha_2} \cdot k_2 + \cdots + \omega^{\alpha_n} \cdot k_n$ where $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ are ordinals and $k_1, k_2, \ldots, k_n$ are nonzero natural numbers. (a). Find the Cantor normal form for $\beta \cdot m$ were m is a nonzero natural number. Prove it. (b). Find the Cantor normal form for $\beta \cdot \omega$. Prove it.

5. Let β be as problem 4. Find the Cantor normal form for $\beta \cdot \omega^\gamma \cdot p$ where γ is an ordinal and p is a nonzero natural number. Prove it. [Note: by the distributive property of multiplication, this lets you find the Cantor normal form of the product of any two ordinals written in Cantor normal form.]

6. Let V be the set of functions $f : \mathbf{N} \to \mathbf{N}$ such that $\{k \in \mathbf{N} : f(k) \neq 0\}$ is finite. Let $<$ be the reverse lexicographic ordering on V, so that $f < g$ if and only if there is an n such that $f(n) < g(n)$ and $f(m) = g(m)$ for all $m > n$. Find an isomorphism from $(V, <)$ to an ordinal. Explain.

7. Suppose that α is a countable limit ordinal. Prove that there is a sequence of ordinals β_n ($n \in \mathbf{N}$) such that

$$\beta_0 \le \beta_1 \le \beta_2 \le \ldots,$$
$$\beta_n < \alpha \quad \text{for each } n, \text{ and}$$
$$\alpha = \sup_{n \in \mathbf{N}} \beta_n.$$

8. Define sets V_α by transfinite recursion as follows:

$$V_0 = \emptyset$$
$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$
$$V_\alpha = \cup_{\beta<\alpha} V_\beta \qquad \text{if } \alpha \text{ is a nonzero limit ordinal.}$$

(a). Prove that each V_α is transitive. (b). [Not to turn in] Prove that $\alpha < \beta$ implies tha $V_\alpha \subset V_\beta$. (c). Suppose that S is a finite set each of whose elements is an element in V_ω. (In other words, suppose that S is a finite subset of V_ω.) Prove that $S \in V_\omega$.

9. Mr. Creosote doesn't believe in any sets except those that are elements of V_ω. (See problem 8.) Thus he means by the word "set" what we mean by the phrase "element of V_ω". Which of the axioms of set theory are true under Mr. Creosote's interpretation of the word "set"? Explain.

[Hint. It might help if you first prove the following: If $S \in V_\omega$, then S is finite.]