MATH 161 HOMEWORK 8 (DUE SATURDAY, MARCH 12)

1. Let κ be an infinite cardinal number. Suppose S is a set such that $|S| \le \kappa$ and such that for each $A \in S$, $|A| \le \kappa$. Prove that
$$|\cup S| \le \kappa.$$
(In other words, if U is the union of at most κ sets each of which has cardinality $\le \kappa$, then U has cardinality $\le \kappa$.)

2. [Not to turn in.] Let α be an ordinal and κ be a cardinal. Show that $|\alpha| < \kappa$ if and only if $\alpha < \kappa$.

3. (In this problem, exponentiation is ordinal exponentiation.) Let κ be an infinite cardinal number. Suppose that α and β are ordinals each of cardinality at most κ. (Thus $|\alpha| \le \kappa$ and $|\beta| \le \kappa$.) Prove that $|\alpha^\beta| \le \kappa$.

Remark: this shows how extremely different ordinal and cardinal exponentiation are. For example, this problem shows that if α and β are countable, then α^β is also countable. On the other hand, if we use cardinal exponentiation, then $\aleph_0^{\aleph_0}$ is of course uncountable.

4. Let $(L, <)$ be a linearly ordered set. Prove that $<$ is a well-ordering of L if and only there is no function $f : \mathbf{N} \to L$ such that $f(n+1) < f(n)$ for every n.

5. Let S be a subset of the plane $\mathbf{R}^2$ such that no three points in S are collinear. Prove that there is a set $T \subset \mathbf{R}^2$ such that (i) $S \subset T$, (ii) no three points in T are collinear, and (iii) if $T \subsetneq V \subset \mathbf{R}^2$, then V contains 3 collinear points.

6. Let S be a collection of sets. We say that S has the **finite intersection property** provided the following holds:
$$\text{if } A \subset S \text{ is finite and nonempty, then } \cap A \text{ is nonempty.}$$

(a). Suppose X is any set and that $S \subset \mathcal{P}(X)$ is a collection of subsets of X such that S has the finite intersection property. Prove there is a set $T \subset \mathcal{P}(X)$ such that (i) $S \subset T$, (ii) T has the finite intersection property, and (iii) if $T \subsetneq V \subset \mathcal{P}(X)$, then V does *not* have the finite intersection property.

(b). Let $A \subset X$. Prove that either $A \in T$ or $X \setminus A \in T$. (Here T is as in part (a).)

(c). Prove that if $A \in T$ and $B \in T$, then $A \cap B \in T$ (where T is as in part (a).)

Example 1: Suppose that $\cap S$ is nonempty. Let $p \in \cap S$. Then we can let $T = \{A \subset X : p \in A\}$.

Example 2: For $n \in \mathbf{N}$, let $A_n = \{k \in \mathbf{N} : k > n\}$, and let $S = \{A_n : n \in \mathbf{N}\}$. Then S has the finite intersection property. In this case, the existence of T is perhaps surprising.