MATH 161 MIDTERM SOLUTIONS

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1. Complete the following definitions.

(a). A is a subset of B provided $\boxed{\text{ every element of } A \text{ is an element of } B}$ or $\boxed{\ x \in A \implies x \in B\ }$.

(b). A strict partial ordering of a set A is $\boxed{\text{a transitive, asymmetric relation on } A}$ or $\boxed{\text{ a relation } R \text{ on } A \text{ such that (i) } xRy \text{ and } yRz \text{ imply } xRz, \text{ and (ii) there is no } x \text{ such that } xRx.}$

(c). If A is a set, $\cup A$ is the set B such that $\boxed{x \in B \text{ if and only if } x \in y \text{ for some } y \in A}$.

(d). A relation R is called **transitive** provided $\boxed{xRy \text{ and } yRz \text{ imply } xRz}$.

(e). Sets A and B are said to be **equipotent** provided $\boxed{\text{there is a bijection from } A \text{ to } B}$.

[not: "A and B have the same cardinality"]

(f). A **relation** from A to B is $\boxed{\text{a subset of } A \times B}$ or $\boxed{\text{a set whose elements are the ordered pairs } (a, b) \text{ with } a \in A \text{ and } b \in B}$.

(g). Let R be a relation from A to B. The inverse relation R^{-1} is $\boxed{\{(x, y) \in B \times A : (y, x) \in R\}}$.

(h). Let R be a relation. We say that R is a **function** provided $\boxed{xRy \text{ and } xRz \text{ imply } y = z}$.

(i). If A and B are sets, then A^B is $\boxed{\text{the set of all functions } f \text{ from } B \text{ to } A}$ or $\boxed{\text{the set of all functions } f \text{ such that } \text{dom}(f) = B \text{ and } \text{ran}(f) \subset A}$.

(j). The **successor** of A is $\boxed{x \cup \{x\}}$.

2. State the axiom [schema] of comprehension (i.e., selection).

Solution: Let $P(x)$ be a sentence. If A is a set, then there is a set B such that

$$x \in B \text{ if and only if } x \in A \text{ and } P(x).$$

3. Let $\mathcal{L}$ be the set of all lines in $\mathbf{R}^2$. Prove that $\mathcal{L} \cong \mathbf{R}$.

Solution: Define a map $F : \mathcal{L} \to \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ by

$$F(L) = \begin{cases} (m, b, 0) & \text{if } L \text{ is the line } y = mx + b, \\ (0, 0, c) & \text{if } L \text{ is the line } x = c. \end{cases}$$

Then F is an injection from $\mathcal{L}$ into $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$, so

(*) $$|\mathcal{L}| \leq |\mathbf{R} \times \mathbf{R} \times \mathbf{R}| = |\mathbf{R} \times \mathbf{R}| = |\mathbf{R}|$$

since we proved that $\mathbf{R} \times \mathbf{R} \cong \mathbf{R}$. Note also that

(**) $$|\mathbf{R}| \leq |\mathcal{L}|$$

since the map

$$G : \mathbf{R} \to \mathcal{L}, \qquad G(c) = \text{ (the line } x = c)$$

is one-to-one. By (*), (**), and Cantor-Bernstein, $|\mathcal{L}| = |\mathbf{R}|$.

Alternate Solution: Let $V \subset \mathcal{L}$ be set of vertical lines, i.e., lines of the form $x = c$. Now $V \cong \mathbf{R}$, since the map

$$f : \mathbf{R} \to V$$
$$f(c) = \text{(the line } x = c)$$

is a bijection. Also, $\mathcal{L} \setminus V \cong \mathbf{R} \times \mathbf{R}$ since the map

$$g : \mathbf{R} \times \mathbf{R} \to \mathcal{L} \setminus V$$
$$g(m, b) = \text{ (the line } y = mx + b)$$

is a bijection. Thus

$$|\mathcal{L}| = |\mathcal{L} \setminus V| + |V| = |\mathbf{R} \times \mathbf{R}| + |\mathbf{R}| = |\mathbf{R}| \cdot |\mathbf{R}| + |\mathbf{R}| = 2^{\aleph_0} \cdot 2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0}.$$

4. Let S be the set of all triangles in $\mathbf{R}^2$ whose vertices have integer coordinates. Prove that S is countable.

Solution: Let T be the set of $(a, b, c) \in (\mathbf{Z}^2) \times (\mathbf{Z}^2) \times (\mathbf{Z}^2)$ such that a, b, and c are not collinear. Then T is countable since $\mathbf{Z}^2 \cong \mathbf{Z} \times \mathbf{Z}$ and since the Cartesian product of two countable sets is countable.

Define a map $F : T \to S$ by letting $F((a, b, c))$ be the triangle with vertices a, b, and c.

Then F is surjective, so S is finite or countable since T is countable. But S is clearly infinite, so S is countable.

5. Let A, B, and C be sets such that B and C are disjoint. Prove that $A^{B \cup C} \cong A^B \times A^C$ by defining maps $F : A^{B \cup C} \to A^B \times A^C$ and $G : A^B \times A^C \to A^{B \cup C}$ such that F and G are inverses of each other (and therefore are both bijections.) [You don't need to prove that they're inverses – just define the maps.]

Solution: $F(u) = (u|B, u|C)$, $G((u, v)) = u \cup v$.

Or G can be described less succinctly: given $(u, v) \in A^B \times A^C$, $G(u, v) : B \cup C \to A$ is the map defined by:

$$G((u, v))(x) = \begin{cases} u(x) & \text{if } x \in B, \\ v(x) & \text{if } x \in C. \end{cases}$$

6. Suppose one interprets "set" to mean "natural number" and one interprets "$\in$" in the usual way. (Thus $n \in m$ if and only if $n < m$.) List the axioms of set theory, and for each axiom, indicate whether it's true or false for this interpretation (and explain). Use the weak forms of the axioms. [You needn't include replacement, choice, or foundation, since those axioms are not in chapters 1-5.]

To avoid confusion, I'll use nset, nsubset, npowerset, etc to indicate what set, subset, powerset, etc, would mean to someone who thinks that "set" means "natural number". Thus a nset is just what we would call a natural number.
Existence: true (e.g. 0).
Extension: true (because the nelements of an nset are the same as its elements.)
Selection: false. For example, $2 = \{0, 1\}$ and $1 = \{0\}$ are nsets (i.e, natural numbers), so if selection were true, then

$$\{x \in 3 : x \notin 1\} \text{ (in other words, } \{2, 1\})$$

would also be a nset (i.e, natural number). But it is not.

Pair: true. Given any two nsets n and m, there is another nset (say $n + m + 1$) that contains n and m as elements.

Note: the strong version of axiom of pairs is false. For example, 2 and 3 are nsets, but any nset that has 2 and 3 as elements will also have the nsets 0 and 1 as elements.

Union: true. Let n be a nset. The axiom of union says that there is a nset u such that

(*) $$x \in y \text{ and } y \in n \text{ implies } x \in u.$$

In the interpretation referred to in this problem, x, y, and z should range over nsets (instead of over all sets), and for natural numbers $\in$ is the same as $<$, so we can rewrite (*) as

$$x < y \text{ and } y < n \text{ implies } x < u..$$

Any number $u \geq n - 1$ has this property. [In fact, the strong form of the axiom of union happens to be true: the nunion of n turns out to be equal to the union of n (namely $n - 1$ if $n > 0$ and 0 if $n = 0$.)

Powerset: True. If n and m are nsets, then n is an nsubset of m if and only if it $n \leq m$. Thus $n + 1$ is the npowerset of n.

Some people thought that the powerset axiom was false in the n-universe. For example, they thought that $3 = \{0, 1, 2\}$ cannot have an npowerset because $\{0, 2\}$ is a subset of 3, but it is not an element of any nset. However, note that

(*) $$(\forall x)(x \subset 3 \implies x \in 4)$$

is true in the n-universe, because in the n-universe, x ranges over all nsets, not over all sets; $x = \{0, 2\}$ is *not* a counterexample to (*) in the n-universe, because it doesn't even exist in the n-universe.

Infinity: "doesn't make sense" or "false" are both acceptable answers:

"Doesn't make sense". One could say successor doesn't make sense, because we defined $S(x) = x \cup \{x\}$, and (in the interpretation suggested in this problem) $\{x\}$ does not makes sense (in general). For example, 5 is a nset, but there is no nset whose only nelement is 5. Since $\{x\}$ doesn't make sense, neither does $x \cup \{x\}$.

"False". One could argue that the nsuccessor of the nset x is the nset y whose elements are x together the nelements of x. By that reasoning, the nsuccessor of a nset x is just its usual successor, namely $x+1$. So the axiom of infinity would say: there is an nset Z such that (i) $0 \in Z$, and (ii) if $x \in Z$, then $S(z) \in Z$. Of course such as Z would have to contain all the natural numbers as elements. Thus there is no such nset Z.

7. Let A be a set. Prove that there is no surjection from A to $\mathcal{P}(A)$.

Solution: Let $F : A \to \mathcal{P}(A)$. Let

$$S = \{x \in A : x \notin F(x)\}.$$

Then S is a subset of A, so $S \in \mathcal{P}(A)$. Let $x \in A$. Then by definition of S,

$$x \in S \text{ if and only if } x \notin F(x).$$

Thus S and $F(x)$ do not have the same elements (since x is in one and not in the other.) Thus $S \neq F(x)$. Since this is true for every $x \in A$, S is not in the range of F, so F is not surjective.

8. (Cardinal arithmetic.) Let f be a function whose domain is $\mathbf{N}$ such that $f(0) = \aleph_0$ and such that $f(n+1) = 2^{f(n)}$ for every $n \in \mathbf{N}$. (Here $f(n)$ is a cardinal number, and $2^{f(n)}$ refers to cardinal exponentiation.) Prove for every n that if $1 \leq \kappa \leq f(n)$, then $\kappa \cdot f(n) = f(n)$.

Solution: First we prove a lemma:

Lemma. *$\aleph_0 \leq f(n)$ for every n.*

Proof of lemma. We prove the lemma by induction on n. Of course $\aleph_0 \leq \aleph_0$, so it is true for $n = 0$. Assume it is true for n: $\aleph_0 \leq f(n)$. Now $f(n+1) = 2^{f(n)} > f(n)$, so $f(n+1) \geq \aleph_0$. $\square$

Now we prove the assertion of the problem by induction on n. The $n = 0$ case was proved in class and in the text, so it suffices to prove that if it is true for $n = m$, then it is true for $n = m+1$. So suppose it is true for $n = m$, and suppose that $1 \leq \kappa \leq f(m+1)$, i.e., that

$$1 \leq \kappa \leq 2^{f(m)}.$$

Multiply by $2^{f(m)}$:

$$2^{f(m)} \leq \kappa \cdot 2^{f(m)} \leq 2^{f(m)} \cdot 2^{f(m)} = 2^{f(m)+f(m)} = 2^{2f(m)} = 2^{f(m)}$$

where $2f(m) = f(m)$ by the induction hypothesis. (Note we need the lemma to know that $1 \leq 2 \leq f(m)$.) $\square$

Remark. Only two people pointed out the need for such a lemma. Of course in this instance, it is rather obvious that $2 \leq f(n)$, and I didn't take off points for failing to prove it.