MATH 161 PRACTICE PROBLEMS FOR FINAL EXAM

1. Let S be any set. Prove that the power set $\mathcal{P}(S)$ of S has greater cardinality than S.

2. Let S be any set. Prove (without using the power set axiom) that there is an x such that $x \notin S$.

3(a). Prove that $\omega \times \omega$ is countable by giving an one-to-one map from $\omega \times \omega$ to ω.

3(b). Let X be the set of all finite subsets of $\mathbf{N}$. Prove that X is countable by giving a one-to-one map from X into $\mathbf{N}$.

4. Let $F : X \to Y$ be a surjective (i.e., "onto") map. Prove that there is an injective (i.e., "one-to-one") map $G : Y \to X$.

5. Define an order relation R on $\mathbf{N}$ by

$$R = \{(a,b) : a < b \text{ and } a - b \text{ is even}\} \cup \{(a,b) : a \text{ is even and } b \text{ is odd}\}$$

(a). Prove that R is a well-ordering of $\mathbf{N}$.

(b). Find an isomorphism from $(\mathbf{N}, R)$ to an ordinal.

6. Define an order relation $<$ on the power set $\mathcal{P}(\omega)$ of ω as follows. If $A \neq B$, let n be the smallest number in $(A \setminus B) \cup (B \setminus A)$. We let $A < B$ if $n \in B$ and $B < A$ if $n \in A$.

Prove or disprove that $<$ is a well-ordering of $\mathcal{P}(\omega)$.

7(a). Let α be an ordinal number. Let $\mathcal{F}$ be the set of all functions $f : \alpha \to \alpha$ such that $\{x \in \alpha : f(x) \neq 0\}$ is finite. Define an order relation $<$ on $\mathcal{F}$ as follows:

$$f < g$$

means that there is an $a \in \alpha$ such that

$$f(a) < g(a) \text{ and } f(x) = g(x) \text{ for all } x > a.$$

(In other words: look at the largest a for which $f(a) \neq g(a)$. Then f and g are in the same order that $f(a)$ and $g(a)$ are.)

Prove that $<$ is a well-ordering of $\mathcal{F}$.

7(b). Let f and g be in $\mathcal{F}$. We say that f is the **predecessor** of g if g is the smallest element greater than f. Which elements of $\mathcal{F}$ do not have predecessors?

BONUS PROBLEM. Let $f : \omega_1 \to \omega_1$ be a function such that $x < y$ implies $f(x) < f(y)$. Prove that there are uncountably many a's such that

$$x < a \implies f(x) < a.$$