MATH 161: SOLUTIONS TO PRACTICE PROBLEMS FOR THE FINAL EXAM

1. Let S be any set. Prove that the power set $\mathcal{P}(S)$ of S has greater cardinality than S.

Solution: The injection $a \in S \mapsto \{a\} \in \mathcal{P}(S)$ shows that $|S| \le |\mathcal{P}(S)|$. There is no bijection (indeed, no surjection) from S to $\mathcal{P}(S)$ because if $F : S \to \mathcal{P}(S)$, then $\{a \in S : a \notin F(a)\}$ is not in the image of F. (See midterm solutions for details.)

2. Let S be any set. Prove (without using the power set axiom) that there is an x such that $x \notin S$.

Solution: Let $T = \{x \in S : x \notin x\}$. Thus for each $x \in S$, $x \in T$ if and only if $x \notin x$. Now suppose $T \in S$. Then $T \in T$ if and only if $T \notin T$, which is a contradiction. Thus $T \notin S$.

3(a). Prove that $\omega \times \omega$ is countable by giving an one-to-one map from $\omega \times \omega$ to ω.

Solution: For example, $f(p, q) = 2^p\, 3^q$.

3(b). Let X be the set of all finite subsets of $\mathbf{N}$. Prove that X is countable by giving a one-to-one map from X into $\mathbf{N}$.

Solution: Let p_n be the nth prime number (so that $p_0 = 2$, $p_1 = 3$, etc.) Now define an injection $F : X \to \mathbf{N}$ by

$$F(\emptyset) = 1$$
$$F(S) = \Pi_{n \in S} p_n \qquad \text{if } S \neq \emptyset$$

Alternate solution Define a bijection(!) $G : X \to \mathbf{N}$ by

$$G(S) = \sum_{n \in S} 2^n.$$

4. Let $F : X \to Y$ be a surjective (i.e., "onto") map. Prove that there is an injective (i.e., "one-to-one") map $G : Y \to X$.

Proof. Let ϕ be a choice function for $\mathcal{P}(X)$. Define $G : Y \to X$ by

$$G(y) = \phi(\{x \in X : F(x) = y\}).$$

Since F is surjective, $\{x \in X : F(x) = y\}$ is nonempty, so $G(y) \in \{x \in X : F(x) = y\}$, so $F(G(y)) = y$. It follows that G is one-to-one. For suppose

$$G(y_1) = G(y_2).$$

Then

$$F(G(y_1)) = F(G(y_2))$$

so $y_1 = y_2$. □

5. Define an order relation R on $\mathbf{N}$ by

$$R = \{(a, b) : a < b \text{ and } a - b \text{ is even}\} \cup \{(a, b) : a \text{ is even and } b \text{ is odd}\}$$

(a). Prove that R is a well-ordering of $\mathbf{N}$.

Proof of Transitivity. Suppose aRb and bRc. We must prove that aRc. Note that is is true if a, b, and c are all even or all odd, since R agrees with $<$ on the even numbers and on the odd numbers.

Case 1: c is odd. If a is even, we are done (since aRc for every even a and odd c.) Thus we may suppose that a is odd. But then b is also odd since aRb. But then a, b, and c are all odd, so transitivity holds.

Case 2: c is even. Then b is even since bRc, and therefore a is even since aRb. Thus a, b, and c are all even, so transitivity holds. □

Proof that aRa can never hold: Suppose aRa. Since $a - a$ is even, this implies that $a < a$, a contradiction. □

Proof of linearity (i.e., that if $a \neq b$, then $a < b$ or $b < a$: Suppose $a \neq b$. If $a - b$ is even, then the smaller of a and b (in the usual ordering $<$) is also smaller in the R ordering. If $a - b$ is odd, then one of a and b is odd and the other one is even. Of course the even one is smaller in the R-ordering than the odd one. □

Proof of well-ordering Suppose S is a nonempty subset of $\mathbf{N}$. If $\{x \in S : x \text{ is even}\}$ is nonempty, then it contains a least element a for the standard ordering $<$. Since the evens precede the odds in the ordering R, a is also the least element in S for the ordering R. If $\{x \in S : x \text{ is even}\}$ is empty, then all elements of S are odd, and so the least element of S for the standard ordering $<$ is also a least element for the ordering R. □

(b). Find an isomorphism from $(\mathbf{N}, R)$ to an ordinal.

Solution: Define $F : \mathbf{N} \to \omega \cdot 2$ by

$$F(2n) = n$$
$$F(2n+1) = \omega + n$$

for every $n \in \mathbf{N}$.

Remark: How do you find the isomorphism? The smallest element of $(\mathbf{N}, R)$, namely 0, has to map to the smallest ordinal, i.e., to 0. The next smallest element of $(\mathbf{N}, R)$, namely 2, has to map to the next smallest ordinal, i.e., 1. Continuing in the way, we see that $F(2n) = n$ for every $n \in \mathbf{N}$. Now we've "used up" the even numbers in $(\mathbf{N}, R)$ and all the finite ordinals (in the range.) So we have to map the first element of $(\mathbf{N}, R)$ after the evens to the first ordinal after all the finite ordinals. That is, $F(1) = \omega$, and so on.

6. Define an order relation $<$ on the power set $\mathcal{P}(\omega)$ of ω as follows. If $A \neq B$, let n be the smallest number in $(A \setminus B) \cup (B \setminus A)$. We let $A < B$ if $n \in B$ and $B < A$ if $n \in A$.

Prove or disprove that $<$ is a well-ordering of $\mathcal{P}(\omega)$.

Solution It is not a well-ordering. Consider, for example, the set S of one-element subsets of ω. Then S is a nonempty subset of $\mathcal{P}(\omega)$, but S has no least element since

$$\{n+1\} < \{n\}.$$

7(a). Let α be an ordinal number. Let $\mathcal{F}$ be the set of all functions $f : \alpha \to \alpha$ such that $\{x \in \alpha : f(x) \neq 0\}$ is finite. Define an order relation $<$ on $\mathcal{F}$ as follows:

$$f < g$$

means that there is an $a \in \alpha$ such that

$$f(a) < g(a) \text{ and } f(x) = g(x) \text{ for all } x > a.$$

(In other words: look at the largest a for which $f(a) \neq g(a)$. Then f and g are in the same order that $f(a)$ and $g(a)$ are.)

Prove that $<$ is a well-ordering of $\mathcal{F}$.

Solution 1: For $f \in \mathcal{F}$, the set $\{x : f(x) > 0\}$ is finite, so it has a greatest element $m(f)$.

Now suppose the ordering is not a well-ordering. Then (see hw8) there is a strictly sequence f_i $(i \in \mathbf{N})$ in $\mathcal{F}$ with $f_1 > f_2 > f_3 > \dots$. Consider the set Z of all $m(f_1)$ where $f_1, f_2, \dots$ is any strictly decreasing sequence in $\mathcal{F}$. Since Z is nonempty, it has a least element x. By definition, there is a sequence $f_1, f_2, \dots$ in $\mathcal{F}$ with $f_1 > f_2 > \dots$ and with $m(f_1) = x$.

Since $f_1 > f_2 > \dots$, we have $x = m(f_1) \geq m(f_2) \geq m(f_3) \geq \dots$. I claim that $m(f_i) = x$ for all i. Proof: if $m(f_i) < x$ (for some i) then $f_i, f_{i+1}, f_{i+2}, \dots$ would be a decreasing sequence in $\mathcal{F}$ with $m(f_i) < x$, contradicting the choice of x. The contradiction proves that $m(f_i) = x$ for all i.

Thus $f_1(x) \geq f_2(x) \geq f_3(x) \geq \dots$. Since α is well-ordered, the set $\{f_1(x), f_2(x), \dots\}$ must have a least element $f_k(x)$. Thus $f_k(x) = f_{k+1}(x) = f_{k+2}(x) = \dots$. We may assume that $k = 1$. (Just drop the first $k-1$ terms in the sequence.)

Now define a new sequence g_i by

$$g_i(t) = \begin{cases} 0 & \text{if } t = x \\ f_i(t) & \text{if } t \neq x \end{cases}$$

Then $g_1 > g_2 > \dots$ and $m(g_1) < x$, contradicting the choice of x. □

Solution 2: More generally, let $\mathcal{F}$ be the set of maps f from α to γ (where α and γ are ordinals) such that $\{x \in \alpha : f(x) \neq 0\}$ is finite. One way to prove that a set $(\mathcal{F}, <)$ is well-ordered is by giving an isomorphism from $(\mathcal{F}, <)$ to some well-ordered set. Since every subset of a well-ordered set is well-ordered, it's enough to give an order-preserving map ϕ from $\mathcal{F}$ **into** some well-ordered set. (In other words, we don't have to check that ϕ is surjective.) In particular, it's enough to define a map from $\mathcal{F}$ to some set of ordinals such that $f < g$ if and only if $\phi(f) < \phi(g)$. We can do this as follows. (Note: it's very similar to the solution to problem 6 of hw7.) If $f \in \mathcal{F}$, we define the ordinal $\phi(f)$ by

$$\phi(f) = \sum_{\{\beta \in \alpha : f(\beta) \neq 0\}} \gamma^\beta \cdot f(\beta) \qquad \text{((decreasing } \beta\text{'s))}.$$

Note that since ordinal addition is not commutative, we have to specify the order of the terms we're adding. Here (as the phrase "decreasing β's" suggests), we order the terms with the highest β first, etc. In other words, if

$$\{\beta \in \alpha : f(\beta) \neq 0\} = \{\beta_1, \beta_2, \ldots, \beta_n\}$$

where $\beta_n > \cdots > \beta_2 > \beta_1$, we let

$$\phi(f) = \gamma^{\beta_n} \cdot f(\beta_n) + \cdots + \gamma^{\beta_2} \cdot f(\beta_2) + \gamma^{\beta_1} \cdot f(\beta_1).$$

With a little work, one can show that $f < g$ if and only if $\phi(f) < \phi(g)$.

(With some more work, one can show that this map is actually an isomorphism from $(\mathcal{F}, <)$ to the ordinal γ^α.)

7(b). Let f and g be in $\mathcal{F}$. We say that f is the **predecessor** of g if g is the smallest element greater than f. Which elements of $\mathcal{F}$ do not have predecessors?

Solution: Let $f \in \mathcal{F}$. The "successor" of f, i.e., the smallest element of $\mathcal{F}$ that is $> f$, is the function g given by

$$g(x) = f(x) \quad \text{if } x \neq 0$$
$$g(0) = f(0) + 1.$$

Thus $g(0) \neq 0$. So if g has a predecessor, then $g(0) \neq 0$.

Conversely, if $g(0) \neq 0$, then g has a predecessor, namely the function f given by

$$f(x) = g(x) \quad \text{if } x \neq 0$$
$$f(0) = g(0) - 1.$$

BONUS PROBLEM. Let $f : \omega_1 \to \omega_1$ be a function such that $x < y$ implies $f(x) < f(y)$. Prove that there are uncountably many a's such that

$$x < a \implies f(x) < a.$$

Solution: Consider the following strategy for the apple game. For each x, if $x < f(x)$, then the set $\{t : x < t \leq f(x)\}$ is finite or countable. Thus there is a one-one map g_x from this set into the set of apples received at stage x.

Here's what you do at stage t of the game. First, if $t < f(x)$ for some $x < t$, consider the smallest such x, and throw away apple $g_x(t)$. If there is no such x, choose any apple to throw away (if you have some) or get fined (if you don't).

Thus you never get fined at times t such that $x < t \leq f(x)$. In other words, you will only get fined at times a such that $x < a$ implies $f(x) < a$. Since you get uncountably many fines (as we know from class), there must be an uncountable set of such a's. □