MATH 161 SOLUTIONS TO SAMPLE MIDTERM QUESTIONS

2. Proof by contradiction. Suppose there is such a set S. By the axiom of selection, there is a set T such that

$$x \in T \text{ if and only if } x \in S \text{ and } x \notin x.$$

Since $x \in S$ for all x, this is equivalent to:

$$x \in T \text{ if and only if } x \notin x.$$

In particular, this is true for $x = T$:

$$T \in T \text{ if and only if } T \notin T,$$

which is clearly a contradiction.

3. Recall that addition of natural numbers is defined so that (i) $n + 0 = n$ and (ii) $n + S(m) = S(n + m)$. Prove from the definition that $0 + n = n$ for every natural number n. [You should prove this "from scratch", i.e., without using facts about addition, subtraction, etc.] **Solution**: We prove by induction that $0 + n = n$ for every n. It holds for $n = 0$ by part (i) of the definition of addition. Now suppose that it holds for $n = k$:

$$(*) \qquad 0 + k = k.$$

We must show that it holds for $n = S(k)$, i.e., that $0 + S(k) = S(k)$. Taking the successor of both sides of (*) gives

$$S(0 + k) = S(k).$$

But $S(0 + k) = 0 + S(k)$ by definition of addition (rule (ii) above), so

$$0 + S(k) = S(k). \quad \square$$

4. For example, $f(n, m) = 2^n 3^m$. Another possibility is $g(n, m) = 2^n(2m + 1) - 1$ (which is actually a bijection).

5. Define $F : 2^{\mathbf{N}} \to \mathcal{P}(\mathbf{N})$ by

$$F(f) = \{x \in \mathbf{N} : f(x) = 1\}.$$

Its inverse is the map $G : \mathcal{P}(\mathbf{N}) \to 2^{\mathbf{N}}$ given by

$$G(A) = 1_A$$

where 1_A is the "indicator function" of A, i.e., the function such that

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \mathbf{N} \setminus A. \end{cases}$$

6. Since $|\mathcal{P}(\mathbf{N})| = |2^{\mathbf{N}}| = 2^{\aleph_0}$ and since $\aleph_0 \cdot \aleph_0 = \aleph_0$, we have

$$|\mathcal{P}(\mathbf{N})^{\mathbf{N}}| = |\mathcal{P}(\mathbf{N})|^{|\mathbf{N}|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = |\mathcal{P}(\mathbf{N})|$$

7. Solution 1: The trick here is to find an "umbrella set" C where $\mathcal{P}(y) \in C$ for every $y \in A$. From there, a simple application of Selection gives us B. Our first impulse is to take $C = \mathcal{P}(A)$, but this doesn't work; not every subset of y need be a element of A, which is what this would require. However, we know that $y \in A$ implies $y \subseteq \bigcup A$, and indeed $z \subseteq \bigcup A$ for every $z \subseteq y$. Thus $z \in \mathcal{P}(\bigcup A)$ for every such z, i.e., $\mathcal{P}(y) \subseteq \mathcal{P}(\bigcup A)$. So $\mathcal{P}(y) \in \mathcal{P}\mathcal{P}(\bigcup A)$, and we take $C = \mathcal{P}\mathcal{P}(\bigcup A)$.

Solution 2 (using axiom of replacement). Let $P(x, y)$ be the sentence "y is the powerset of x". Then for every x, there is a unique y such that $P(x, y)$. The strong form of the axiom of replacement then says that there is a set B such that

$$y \in B \text{ if and only if } y = \mathcal{P}(x) \text{ for some } x \in A.$$

8. Existence: Certainly true. We even have an empty p-set O, if you prefer the stronger version of the axiom.

Extensionality: True. We have two p-sets, one has p-elements, the other does not.

Selection: False. There is no p-set $\{O\} = \{\, x \boxed{\in} I : x = O \,\}$. (Also, if Selection were true, we could use the universal p-set I to create a Russell p-set $B = \{\, x : x \boxed{\notin} x \,\} = \{\, x \in I : x \boxed{\notin} x \,\}$)

Pairing: False. As noted above, there is no p-set $\{O\} = \{O, O\}$. (Remember, nothing says that a and b in $\{a, b\}$ must be distinct!)

Union: True. $\bigcup O = O$, $\bigcup I = I$.

Powerset: False. There is no $\{O\} = \mathcal{P}(O)$.

Foundation (which we haven't officially covered yet): True. The only nonempty p-set is I, and $O \boxed{\in} I$ where O and I have no common p-elements.

(Remark: the axiom of infinity is problematic because because the successor operation is not appropriately defined.)

9. Suppose $a \in \mathcal{P}(S)$; we must show that $a \subset \mathcal{P}(S)$.

Since $a \in \mathcal{P}(S)$,

$$a \subset S. \qquad (*)$$

To show that $a \subset \mathcal{P}(S)$, assume $x \in a$; we must then show that $x \in \mathcal{P}(S)$.

Since $x \in a$, $x \in S$ by (*). Thus $x \subset S$ since S is transitive. Thus $x \in \mathcal{P}(S)$ (by definition of power set.) $\square$