THE CANTOR-BERNSTEIN THEOREM

Theorem (Cantor-Bernstein Theorem)**.** *Suppose that $f : A \to B$ and $g : B \to A$ are injective maps. Then there is a bijection from A to B.*

Idea of proof: We try find a bijection $H : A \to B$ such that for each $x \in A$,

(*) $$\text{either } H(x) = f(x) \text{ or } H(x) = g^{-1}(x).$$

Now suppose $x_0 \in A \setminus g[B]$. Then $g^{-1}(x_0)$ does not exist, so (*) forces us to let

$$H(x_0) = f(x_0).$$

Now let $x_1 = (g \circ f)(x_0) = g(f(x_0))$. Since $x_1 \in \text{ran}(g)$ and $x_0 \notin \text{ran}(g)$, so $x_0 \neq x_1$. We can't let $H(x_1) = g^{-1}(x_1)$, because then H would map both x_0 and x_1 to $f(x_0)$. Thus we are forced to let $H(x_1) = f(x_1)$. Similarly, if we let $x_n = (g \circ f)^n(x_0)$, we are forced to let $H(x_n) = f(x_n)$. These considerations suggest the following proof.

Proof. Define sets C_n recursively as follows:

(1) $C_0 = A \setminus g[B]$.
(2) $C_{n+1} = g[f[C_n]]$.

Let $C = \cup_{n \in \mathbf{N}} C_n$. Since g is one-to-one, its inverse g^{-1} is also a function. Note that $x \in A \setminus C$ implies that $x \in A \setminus C_0 = g[B]$. Thus

$$A \setminus C \subset \text{dom}(g^{-1}).$$

Now define a map $H : A \to B$ by:

$$H(x) = \begin{cases} f(x) & \text{if } x \in C, \\ g^{-1}(x) & \text{if } x \in A \setminus C. \end{cases}$$

First we prove that $H : A \to B$ is onto. Let $y \in B$. Either $g(y) \in C$ or $g(y) \in A \setminus C$.

Case 1: If $g(y) \in C$, then $g(y) \in C_n$ for some $n \in \mathbf{N}$. That n cannot be 0, because $C_0 = A \setminus g[B]$. Thus $g(y) \in C_{k+1}$ for some $k \in \mathbf{N}$, so

$$g(y) = g(f(x)) \text{ for some } x \in C_k.$$

Since g is one-to-one, this implies that $y = f(x)$. Since $x \in C$, $f(x) = H(x)$, so $y = H(x)$ and thus $y \in \text{ran}(H)$.

Case 2: If $g(y) \notin C$, then $H(g(y)) = g^{-1}(g(y)) = y$, so $y \in \text{ran}(H)$.

We have shown that each $y \in B$ is in the range of H, and thus that $H : A \to B$ is onto.

To show that H is one-to-one, suppose that $H(x) = H(y)$. We must show that $x = y$. There are three cases:

Case 1: $x, y \in C$. Then $H(x) = f(x)$ and $H(y) = f(y)$, so $f(x) = f(y)$ and therefore $x = y$ since f is one-to-one.

Case 2: $x, y \in A \setminus C$. Then $H(x) = g^{-1}(x)$ and $H(y) = g^{-1}(y)$, so $g^{-1}(x) = g^{-1}(y)$ and therefore $x = g(g^{-1}(x)) = g(g^{-1}(y)) = y$.

Case 3: One of x and y is in C and the other is in $A \setminus C$. We may suppose $x \in C$ and $y \in A \setminus C$. Then $H(x) = f(x)$ and $H(y) = g^{-1}(y)$, so

$$f(x) = g^{-1}(y)$$

and therefore

$$g(f(x)) = y.$$

Since $x \in C$, $g(f(x)) \in C$, so $y \in C$, which is a contradiction. (Thus case 3 cannot occur.) □

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