MATH 161 LECTURE NOTES: BASIC FACTS ABOUT ORDINAL ARITHMETIC

Here we summarize the basic facts about ordinal arithmetic:

(1) $\alpha + 0 = 0 + \alpha = \alpha$.
(2) $\alpha + 1 = S(\alpha)$.
(3) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
(4) $\alpha + \sup_{\beta \in B} \beta = \sup_{\beta \in B}(\alpha + \beta)$.
(5) $k + \omega = \omega$ for all $k < \omega$.
(6) $\alpha \cdot 0 = 0 \cdot \alpha = 0$.
(7) $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$.
(8) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
(9) $\alpha \cdot \sup_{\beta \in B} \beta = \alpha \cdot \sup_{\beta \in B}$.
(10) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
(11) $k \cdot \omega = \omega$ if $0 < k < \omega$.
(12) $\alpha^0 = 1$.
(13) $\alpha^1 = \alpha$.
(14) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.
(15) $\alpha^{\sup_{\beta \in B} \beta} = \sup_{\beta \in B} \alpha^\beta$.
(16) $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.
(17) If $\alpha \geq \beta$, then there is a unique ordinal $\gamma = \alpha - \beta$ such that $\beta + \gamma = \alpha$.

Additional facts that involve inequalities:

(i) If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.
(ii) $\alpha \leq \beta + \alpha$.
(iii) If $\beta < \gamma$ and $\alpha \geq 1$, then $\alpha \cdot \beta < \alpha \cdot \gamma$, and $\beta \leq \alpha \cdot \beta$.
(iv) If $\beta < \gamma$ and $\alpha \geq 2$, then $\alpha^\beta < \alpha^\gamma$, and $\beta \leq \alpha^\beta$.

Lemma 1. *If α is any ordinal and β is a nonzero ordinal, then there is a unique pair of ordinals γ and δ such that*

$$\alpha = \beta \cdot \gamma + \delta \text{ and } \delta < \beta.$$

If α is any nonzero ordinal and $\beta \geq 2$, then there is a largest ordinal γ such that

$$\beta^\gamma \leq \alpha.$$

Furthermore, there is a nonzero $k < \omega$ and an ordinal $\delta < \beta^\gamma$ such that

$$\alpha = \beta^\gamma \cdot k + \delta.$$

Theorem 2 (Cantor Normal Form)**.** *Let α be a nonzero ordinal. Then exist a nonzero natural number n, ordinals $\beta_1 > \beta_2 > \cdots > \beta_n$ and nonzero natural numbers $k_1, k_2, \ldots, k_n$ such that*

$$(*) \qquad \alpha = \sum_{i=1}^{n} \omega^{\beta_i} \cdot k_i$$

Furthermore, the β_i's and k_i's are uniquely determined by α.

Note: since addition is not commutative, the order in (*) is important! We define $\sum_{i=1}^n$ so that the first term is left most:

$$\sum_{i=1}^{n} \omega^{\beta_i} \cdot k_i = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n.$$

We did not show uniqueness in class, but the uniqueness is a consequence of the following proposition, which shows how to determine which of two ordinals in Cantor normal form is the smaller one.

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Proposition 3. *Suppose*

$$\alpha = \sum_{i=1}^{n} \omega^{\beta_i} \cdot k_i$$
$$\alpha' = \sum_{i=1}^{n} \omega^{\beta_i} \cdot k_i'.$$

where $\beta_1 > \beta_2 > \cdots > \beta_n$ and where the k_i and k_i' are natural numbers (possibly zero). Suppose also that there is an m with $1 \le m \le n$ such that

$$k_m < k_m', \text{ and}$$
$$k_i = k_i' \text{ for all } i < m.$$

Then $\alpha < \alpha'$.

Note that in proposition ? we are allowing some of the k_i and k_i' to be 0. This is so that we can express α and α' using the same exponents $\beta_1, \dots, \beta_n$.

Lemma 4. (1) *If $\beta > \gamma$ and if n and m are natural numbers, then*

$$\omega^\beta \cdot m + \omega^\gamma \cdot n < \omega^\beta \cdot (m+1).$$

(2) *If $\beta_1 > \beta_2 > \cdots > \beta_n$ and if n, $k_1, \dots, k_n$ are natural numbers, then*

$$\sum_{i=1}^{n} \omega^{\beta_i} \cdot k_i < \omega^{\beta_1} \cdot (k_1 + 1).$$

Proof. To prove (1), note that $n < \omega$, so

$$\omega^\gamma \cdot n < \omega^\gamma \cdot \omega = \omega^{\gamma+1} \le \omega^\beta.$$

Thus

$$\omega^\beta \cdot m + \omega^\gamma \cdot n < \omega^\beta \cdot m + \omega^\beta = \omega^\beta \cdot (m+1).$$

Assertion (2) follows from (1) by induction on n. □

Proof of proposition 3. By part (2) of the lemma,

$$\begin{aligned} \sum_{i \ge m} \omega^{\beta_i} \cdot k_i &< \omega^{\beta_m} \cdot (k_m + 1) \\ &\le \omega^{\beta_m} \cdot k_m' \qquad \text{(since } k_m < k_m'\text{)} \\ &\le \sum_{i \ge m} \omega^{\beta_i} \cdot k_i', \end{aligned}$$

so

$$(*) \qquad \sum_{i \ge m} \omega^{\beta_i} \cdot k_i < \sum_{i \ge m} \omega^{\beta_i} \cdot k_i'.$$

The proposition follows since $\eta < \nu$ implies $\delta + \eta < \delta + \nu$. (Add $\delta = \sum_{i=1}^{m-1} \omega^{\beta_i} \cdot k_i$ on the left to both sides of (*).) □