MATH 161 LECTURE NOTES: WELL-ORDERED SETS AND ORDINALS

These notes give the proof of a fundamental fact about well-ordered sets, namely

Theorem 1. *Let $(W, <)$ be a well-ordered set. Then there is a unique ordinal isomorphic to $(W, <)$.*

The proof is based on two principles: the axiom of replacement, and the following theorem (which we proved earlier):

Theorem 2. *Given any two well-ordered sets, either they are isomorphic, or else one is isomorphic to an initial segment of the other.*

Before giving the proof of theorem 1, we recall the axiom of replacement. Here $P(x, y)$ is a sentence involving x and y.

Axiom of replacement. *Suppose that for every x, there is a unique y such that $P(x, y)$. Let A be a set. Then there is a set B such that*

$$y \in B \text{ if and only if } P(x, y) \text{ for some } x \in A.$$

It may seem peculiar that the hypothesis involves all x's, whereas the conclusion only involves x's in A. Intuitively, if we only care about $x \in A$, then whether $P(x, y)$ is true or false for x not in A should be irrelevant. Indeed, that is the case:

Theorem (Axiom of replacement, alternate version)**.** *Let A be a set. Suppose that for every $x \in A$, there is a unique y such that $P(x, y)$. Then there is a set B such that*

(*) $$y \in B \text{ if and only if } P(x, y) \text{ for some } x \in A.$$

Proof. Let $Q(x, y)$ be the sentence "Either $x \in A$ and $P(x, y)$, or $x \notin A$ and $y = 0$". Now (*) implies that for *every* x, there is a unique y such that $Q(x, y)$. Thus by the axiom of replacement, there is a set B such that

(†) $$y \in B \text{ if and only if } Q(x, y) \text{ for some } x \in A.$$

But for $x \in A$, $P(x, y)$ is equivalent to $Q(x, y)$. Thus (*) and (†) are equivalent. □

Now we give the proof of theorem 1:

Proof. Let $P(x, y)$ be the sentence "$x \in W$, y is an ordinal, and y is isomorphic to the initial segment $W[x]$". Let

$$A = \{x \in W : x \text{ is isomorphic to some ordinal}\}.$$

Note that for every $x \in A$, there exists a unique y such that $P(x, y)$. (Existence is by definition of A, and uniqueness is because no two distinct ordinals are isomorphic to each other.) Thus by the alternate version of the axiom of replacement, there is a set B such that

$$y \in B \text{ if and only if } P(x, y) \text{ for some } x \in A.$$

Equivalently,

$$y \in B \text{ if and only if: } y \text{ is an ordinal and } y \text{ is isomorphic to an initial segment of } W.$$

Now B is a set of ordinals, and we know that there is no set containing all ordinals[1]. Thus there is an ordinal α that is not an element of B.

By theorem 2,

(1) $(W, <)$ is isomorphic to α, or
(2) $(W, <)$ is isomorphic to an initial segment of α, or
(3) α is isomorphic to an initial segment of $(W, <)$.

Now in case (1) we are done. In case (2), we are also done because any initial segment of an ordinal is also an ordinal.

But case (3) is impossible, because if α were isomorphic to an initial segment of $(W, <)$, then α would be an element of B, but α was chosen to be an ordinal not in B. □

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[1]To see that B does not contain every ordinal, recall that $\cup B$ is an ordinal that is greater than or equal to every ordinal in B, and thus $S(\cup B)$ is an ordinal that is strictly greater than every ordinal in B.