THE FIRST FEW ORDINALS

The first ordinal numbers are the natural numbers: $0, 1, 2, \ldots$. The next one is ω (i.e., $\mathbf{N}$), followed by $S(\omega)$, $S(S(\omega))$, $\ldots$. These can also be written as

$$\omega,\ \omega+1,\ \omega+2,\ \omega+3, \ldots$$

Next comes $\omega \cdot 2$, followed by $\omega \cdot 2 + 1$, $\omega \cdot 2 + 2$, $\omega \cdot 3 + 3, \ldots$.

Continuing in this way, we get ordinals of the form

$$\omega \cdot n + m$$

for every pair of natural numbers n and m. The ordering of these is as follows: if a, b, c, d are natural numbers with $a < c$, then $\omega \cdot a + b < \omega \cdot c + d$, and if $b < d$, then $\omega \cdot a + b < \omega \cdot a + d$.

After all those ordinals comes ω^2 and (more generally) all ordinals of the form $\omega^2 + \omega \cdot n + m$, where n and m are natural numbers.

After all those comes $\omega^2 \cdot 2$.

In general, if $\langle a_0, a_1, \ldots, a_n \rangle$ is any finite sequence of natural numbers with $a_n \neq 0$, we can form an ordinal

$$\alpha = \omega^n \cdot a_n + \omega^{n-1} \cdot a_{n-1} + \cdots + \omega^2 \cdot a_2 + \omega \cdot a_1 + a_0.$$

Given another such ordinal

$$\beta = \omega^m \cdot b_m + \omega^{m-1} \cdot b_{m-1} + \cdots + \omega^2 \cdot b_2 + \omega \cdot b_1 + b_0.$$

The ordering is as follows (assuming $a_n \neq 0$ and $b_m \neq 0$): if $n < m$, then $\alpha < \beta$.

If $\alpha \neq \beta$ and $n = m$, let k be the largest number such that $a_k \neq b_k$. Then $\alpha < \beta$ if $a_k < b_k$, and $\beta < \alpha$ if $b_k < a_k$.

After all those ordinals comes ω^ω, but perhaps that's enough for now...

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